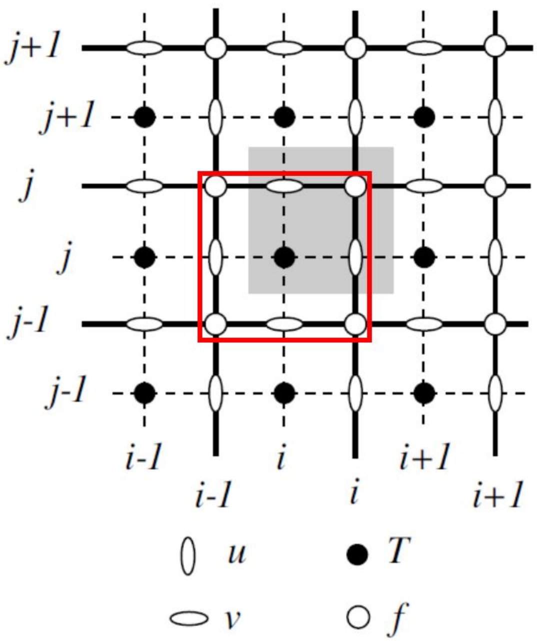
Rotating U and V components of velocity from model grid to a lat-lon grid

The figure below shows the Arakawa-C grid viewed from above. The solid thick lines delimit the grid cells. The indexing of the points is shown by the i and j's. The grey shade includes the T-, u-, v- and f-points that have the same indexing (ie, $T(i,j), u(i,j), v(i,j), f(i,j)$).



To start using the velocities, you should move U and V velocities to the same point; T-point is ideal in case you want to use temperature and salinity as well (already located at T-points).

Let's look at the grid cell highlighted in red in the figure above. Note that:

- there is one u-point at each side of $T(i,j)$: $u(i-1,j)$ and $u(i,j)$;
- and one v-point above and one below $T(i,j)$: $v(i,j)$ and $v(i,j-1)$;

To move the U velocity component from the u-point (U_u) to a T-point (U_T), you should average the U's at each side of T; in this case:

$$U_T\,(i,j) \;=\; \frac{(U_u(i-1,j) \;+\; U_u(i,j)\,)}{2}$$

To move the V velocity component from the v-point (V_v) to a T-point (V_T), you should average the V's above and below T; in this case:

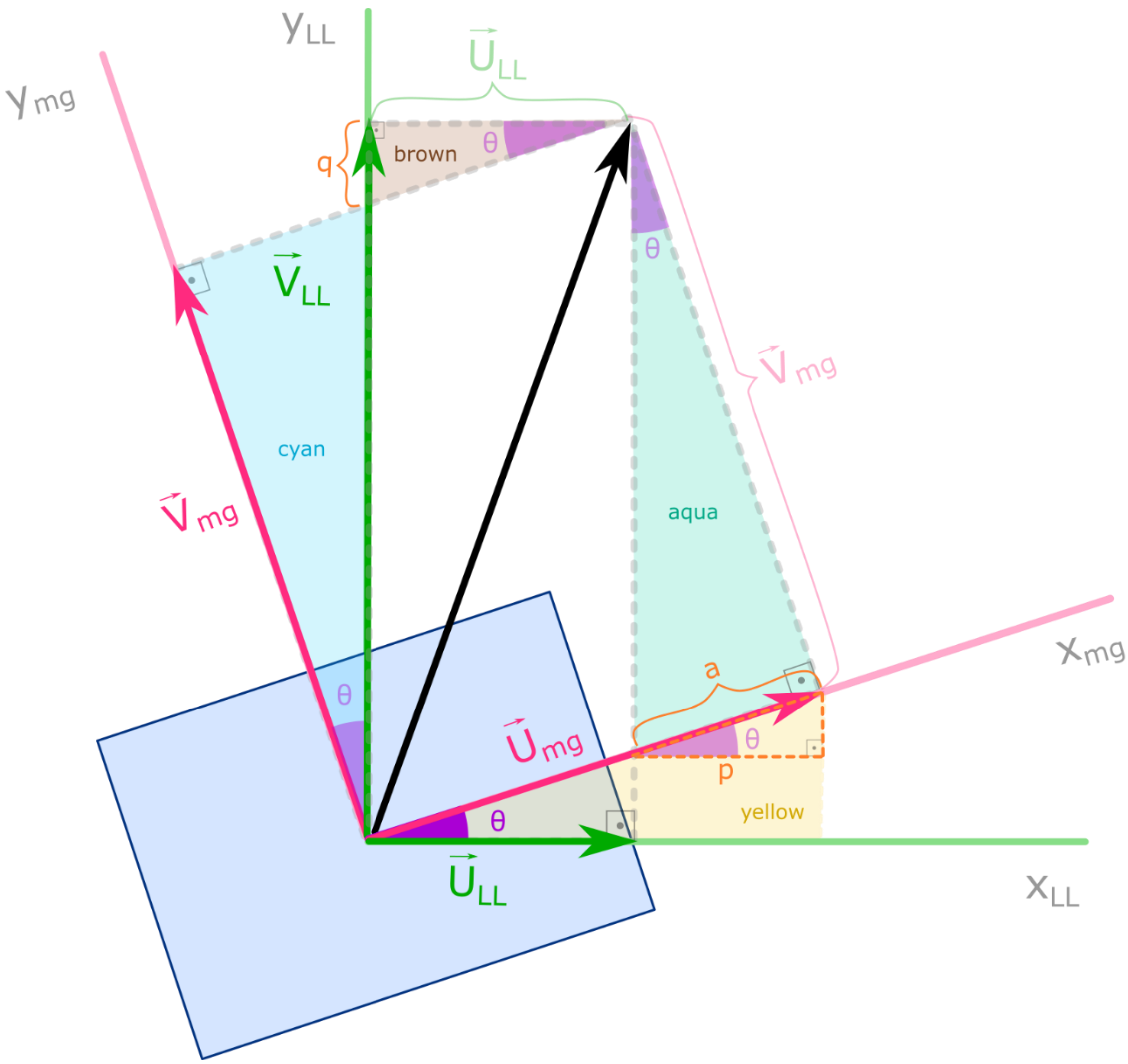
$$V_T\,(i,j) \;=\; \frac{(V_v(i,j) \;+\; V_v(i,j-1)\,)}{2}$$

Now that both components of the velocity are at the same point, we can rotate them. For clarity, we will now call U_T and V_T as $\vec{U}_{mg}$ and $\vec{V}_{mg}$.

The figure below is a schematic of a model grid cell (in dark blue), the velocity vector in black, the model grid x and y orientation (x_{mg}, y_{mg} in magenta), U and V components of velocity in the model grid ($\vec{U}_{mg}, \vec{V}_{mg}$ in magenta), lat-lon grid orientation (x_{LL}, y_{LL} in green), rotated U and V components of the velocity ($\vec{U}_{LL}, \vec{V}_{LL}$ in green, what we want to find), and the angle θ between the model x (x_{mg}) and lat-lon x (x_{LL}), in dark purple.

Note that below we will only deal with the magnitude of the vectors, where

$$\vec{U}_{LL} = U_{LL} \cdot \hat{\imath}_{LL} \qquad \vec{V}_{LL} = V_{LL} \cdot \hat{\jmath}_{LL}$$
$$\vec{U}_{mg} = U_{mg} \cdot \hat{\imath}_{mg} \qquad \vec{V}_{mg} = V_{mg} \cdot \hat{\jmath}_{mg}$$



First, let's find an expression for U_{LL}.

Looking at the yellow triangle, we can see that

$\cos\theta = \frac{U_{LL}+p}{U_{mg}}$ or $U_{LL} = U_{mg} \cdot \cos\theta - p$ (1)

To find p we first look at the triangle delineated by the dashed orange lines. From this triangle, we see that

$$\cos\theta = \frac{p}{a} \quad \text{or} \quad p = a \cdot \cos\theta \tag{2}$$

From the aqua triangle,

$$\tan\theta = \frac{a}{V_{mg}} \quad \text{or} \quad a = V_{mg} \cdot \frac{\sin\theta}{\cos\theta} \tag{3}$$

Substituting (3) in (2),

$$p = V_{mg} \cdot \frac{\sin\theta}{\cos\theta} \cdot \cos\theta \quad \text{or} \quad p = V_{mg} \cdot \sin\theta \tag{4}$$

and (4) in (1)

$$U_{LL} = U_{mg} \cdot \cos\theta - V_{mg} \cdot \sin\theta \tag{5}$$

Great! Now let's find an expression for V_{LL}.

Looking at the cyan triangle,

$$\cos\theta = \frac{V_{mg}}{V_{LL} - q} \quad \text{or} \quad V_{LL} = \frac{V_{mg}}{\cos\theta} + q \tag{6}$$

Now, from the brown triangle,

$$\tan\theta = \frac{q}{U_{LL}} \quad \text{or} \quad q = U_{LL} \cdot \frac{\sin\theta}{\cos\theta} \tag{7}$$

Substituting the U_{LL} expression (5) into (7),

$$q = \left(U_{mg} \cdot \cos\theta - V_{mg} \cdot \sin\theta\right) \cdot \frac{\sin\theta}{\cos\theta}$$

$$q = U_{mg} \cdot \sin\theta - V_{mg} \cdot \frac{\sin^2\theta}{\cos\theta} \tag{8}$$

Replacing (8) into (6), we have

$$V_{LL} = \frac{V_{mg}}{\cos\theta} + U_{mg} \cdot \sin\theta - V_{mg} \cdot \frac{\sin^2\theta}{\cos\theta}$$

$$V_{LL} = V_{mg}\left(\frac{1 - \sin^2\theta}{\cos\theta}\right) + U_{mg} \cdot \sin\theta$$

We know that $\sin^2\theta + \cos^2\theta = 1$, so replacing $1 - \sin^2\theta$ by $\cos^2\theta$, we finally get

$$V_{LL} = V_{mg} \cdot \cos\theta + U_{mg} \cdot \sin\theta \tag{9}$$